

# HERMITIAN-EINSTEIN CONNECTIONS ON POLYSTABLE ORTHOGONAL AND SYMPLECTIC PARABOLIC HIGGS BUNDLES

INDRANIL BISWAS AND MATTHIAS STEMMLER

ABSTRACT. Let  $X$  be a smooth complex projective curve and  $S \subset X$  a finite subset. We show that an orthogonal or symplectic parabolic Higgs bundle on  $X$  with parabolic structure over  $S$  admits a Hermitian-Einstein connection if and only if it is polystable.

## 1. INTRODUCTION

Let  $X$  be an irreducible smooth complex projective curve, and let  $S \subset X$  be a fixed finite subset. The notion of parabolic vector bundles on  $X$  with  $S$  as the parabolic divisor was introduced by Seshadri [Se77]. Parabolic bundles equipped with Higgs fields were introduced by Simpson [Si90] under the name of filtered regular Higgs bundles; see also [Yo93].

An orthogonal or symplectic parabolic bundle is a parabolic vector bundle equipped with a symmetric or alternating form, respectively, with values in a parabolic line bundle; this form is required to be non-degenerate in a suitable sense [BMW11]. In the case of rational parabolic weights, this coincides with the notion of parabolic principal  $G$ -bundles introduced in [BBN01] and [BBN03], where  $G$  is the orthogonal or symplectic group, respectively.

In [BMW11], orthogonal and symplectic parabolic bundles were investigated. In particular, a Hitchin-Kobayashi correspondence, which says that an orthogonal or symplectic parabolic bundle admits a Hermitian-Einstein connection if and only if it is polystable, was established. Our aim here is to define Higgs fields on orthogonal and symplectic parabolic bundles and to generalize the Hitchin-Kobayashi correspondence to this context. We obtain the following result (see Theorem 16 and Proposition 17):

**Theorem 1.** *Let  $(E_*, \varphi, \theta)$  be an orthogonal or symplectic parabolic Higgs bundle. If  $(E_*, \varphi, \theta)$  is polystable, then it admits a Hermitian-Einstein connection.*

*Conversely, if  $(E_*, \varphi, \theta)$  admits a Hermitian-Einstein connection lying in the space  $\mathcal{A}$  (see (4.3)), then it is polystable.*

---

2000 *Mathematics Subject Classification.* 53C07, 14H60.

*Key words and phrases.* Hermitian-Einstein connection, orthogonal and symplectic bundle, parabolic Higgs bundle.

## 2. ORTHOGONAL AND SYMPLECTIC PARABOLIC HIGGS BUNDLES

**2.1. Parabolic vector bundles.** Let  $X$  be an irreducible smooth complex projective curve. Fix a finite subset

$$S := \{x_1, \dots, x_n\} \subset X$$

with distinct points  $x_1, \dots, x_n$  of  $X$ . Let  $E$  be a holomorphic vector bundle on  $X$ . Recall that a *quasi-parabolic structure* on  $E$  over  $S$  is a filtration of subspaces

$$(2.1) \quad E_{x_i} = F_{i,1} \supsetneq \dots \supsetneq F_{i,j} \supsetneq \dots \supsetneq F_{i,\ell_i} \supsetneq F_{i,\ell_i+1} = 0$$

over each point  $x_i$  of  $S$ . A *parabolic structure* on  $E$  over  $S$  is a quasi-parabolic structure as above together with real numbers

$$0 \leq \alpha_{i,1} < \dots < \alpha_{i,j} < \dots < \alpha_{i,\ell_i} < 1,$$

which are called the *parabolic weights*. The weight  $\alpha_{i,j}$  corresponds to the subspace  $F_{i,j}$ . (See [Se77], [Se82, p. 67], [MY92].) A *parabolic vector bundle* with parabolic divisor  $S$  is a holomorphic vector bundle  $E$  equipped with a quasi-parabolic structure over  $S$  and parabolic weights as above.

For convenience, a parabolic vector bundle  $(E, \{F_{i,j}\}, \{\alpha_{i,j}\})$  will be denoted by  $E_*$ .

We fix the divisor  $S$  once and for all. Henceforth, the parabolic divisor for all parabolic vector bundles will be this  $S$ .

The *parabolic degree* of a parabolic vector bundle  $E_*$  is defined to be

$$\text{par-deg}(E_*) := \text{degree}(E) + \sum_{i=1}^n \sum_{j=1}^{\ell_i} \alpha_{i,j} \cdot \dim(F_{i,j}/F_{i,j+1})$$

and the real number

$$\text{par-}\mu(E_*) := \frac{\text{par-deg}(E_*)}{\text{rank}(E)}$$

is called the *parabolic slope* of  $E_*$ .

See [Bi97], [Yo95] for tensor product, dual and homomorphism bundles for parabolic bundles.

**2.2. Orthogonal and symplectic structures.** Fix a parabolic line bundle  $L_*$ . The underlying holomorphic line bundle will be denoted by  $L$ .

Let  $E_*$  be a parabolic vector bundle, and let

$$\varphi : E_* \otimes E_* \longrightarrow L_*$$

be a homomorphism of parabolic bundles. Tensoring both sides of this homomorphism with the parabolic dual  $E_*^*$ , we obtain a homomorphism

$$\varphi \otimes \text{id} : E_* \otimes E_* \otimes E_*^* \longrightarrow L_* \otimes E_*^*.$$

Note that the sheaf of sections of the vector bundle underlying  $E_* \otimes E_*^*$  is the sheaf of endomorphisms of  $E$  preserving the quasi-parabolic filtrations. The trivial line bundle  $\mathcal{O}_X$  equipped with the trivial parabolic structure (meaning there is no non-zero parabolic weight) is realized as a parabolic subbundle of  $E_* \otimes E_*^*$  by sending any locally defined

function  $f$  to the locally defined endomorphism of  $E$  given by pointwise multiplication with  $f$ . Let

$$(2.2) \quad \tilde{\varphi} : E_* \longrightarrow L_* \otimes E_*^*$$

be the homomorphism defined by the composition

$$E_* = E_* \otimes \mathcal{O}_X \hookrightarrow E_* \otimes (E_* \otimes E_*^*) = (E_* \otimes E_*) \otimes E_*^* \xrightarrow{\varphi \otimes \text{id}} L_* \otimes E_*^*.$$

**Definition 2.**

- (i) An *orthogonal parabolic bundle* is a pair  $(E_*, \varphi)$  of the above form such that  $\varphi$  is symmetric, and the homomorphism  $\tilde{\varphi}$  in (2.2) is an isomorphism.
- (ii) A *symplectic parabolic bundle* is a pair  $(E_*, \varphi)$  of the above form such that  $\varphi$  is anti-symmetric, and the homomorphism  $\tilde{\varphi}$  is an isomorphism.

Let  $(E_*, \varphi)$  be an orthogonal or symplectic parabolic bundle with  $E$  as the underlying vector bundle. Then  $E \otimes E$  is a coherent subsheaf of the vector bundle underlying the parabolic tensor product  $E_* \otimes E_*$ . Therefore,  $\varphi$  produces a homomorphism

$$(2.3) \quad \hat{\varphi} : E \otimes E \longrightarrow L,$$

where  $L$  is the holomorphic line bundle underlying  $L_*$ . A holomorphic subbundle

$$F \subset E$$

is called *isotropic* if

$$(2.4) \quad \hat{\varphi}(F \otimes F) = 0,$$

where  $\hat{\varphi}$  is constructed in (2.3).

**2.3. Higgs fields.** Let  $\Omega_X$  be the canonical line bundle of  $X$ . For notational convenience, we write  $\Omega_X(S) := \Omega_X \otimes \mathcal{O}_X(S)$ . Let  $E$  be a holomorphic vector bundle on  $X$ . A *logarithmic Higgs field* on  $E$  is a holomorphic section

$$\theta \in H^0(X, \text{End}(E) \otimes \Omega_X(S)).$$

For every point  $x_i \in S$ , the fiber  $(\Omega_X(S))_{x_i}$  is identified with  $\mathbb{C}$  using the Poincaré adjunction formula. The endomorphism

$$E_{x_i} \xrightarrow{\theta(x_i)} (E \otimes \Omega_X(S))_{x_i} = E_{x_i}$$

is called the *residue* of  $\theta$  at  $x_i$ ; it will be denoted by  $\text{Res}(\theta, x_i)$ .

**Definition 3.**

- (i) Let  $E_*$  be a parabolic vector bundle on  $X$ . A *parabolic Higgs field* on  $E_*$  is a logarithmic Higgs field

$$\theta \in H^0(X, \text{End}(E) \otimes \Omega_X(S))$$

such that for every point  $x_i \in S$ , the residue  $\text{Res}(\theta, x_i)$  preserves the quasi-parabolic filtration in the sense that

$$\text{Res}(\theta, x_i)(F_{i,j}) \subset F_{i,j} \quad \text{for all } 1 \leq j \leq \ell_i$$

(see (2.1)).

- (ii) A *parabolic Higgs bundle* is a pair  $(E_*, \theta)$  consisting of a parabolic vector bundle  $E_*$  and a parabolic Higgs field  $\theta$  on  $E_*$ .

**Lemma 4.** *Let  $E_*$  and  $F_*$  be parabolic vector bundles equipped with parabolic Higgs fields  $\theta_E$  and  $\theta_F$ , respectively. Then  $\theta_E$  and  $\theta_F$  together induce a parabolic Higgs field on the parabolic tensor product  $E_* \otimes F_*$ . Also,  $\theta_E$  induces a parabolic Higgs field on the parabolic dual  $E_*^*$ .*

*Proof.* The logarithmic Higgs field  $\theta_E$  on the vector bundle  $E$  underlying  $E_*$  induces a logarithmic Higgs field on the dual vector bundle  $E^*$ ; this logarithmic Higgs field on  $E^*$  will be denoted by  $\theta'_E$ . Let  $E_0^*$  be the vector bundle underlying the parabolic dual  $E_*^*$ . Then  $E_0^*$  is a subsheaf of  $E^*$ . It is straightforward to check that the logarithmic Higgs field  $\theta'_E$  on  $E^*$  produces a logarithmic Higgs field on  $E_0^*$ . This logarithmic Higgs field on  $E_0^*$  is a parabolic Higgs field on the parabolic vector bundle  $E_*^*$ .

Let  $F$  be the vector bundle underlying  $F_*$ . The two logarithmic Higgs fields  $\theta_E$  and  $\theta_F$  on  $E$  and  $F$  respectively together induce a logarithmic Higgs field on the vector bundle  $E \otimes F \otimes \mathcal{O}_X(S)$  (the Higgs field on  $\mathcal{O}_X(S)$  is taken to be the zero section). The vector bundle  $(E_* \otimes F_*)_0$  underlying the parabolic tensor product  $E_* \otimes F_*$  is a subsheaf of  $E \otimes F \otimes \mathcal{O}_X(S)$ . It is straightforward to check that the above logarithmic Higgs field on  $E \otimes F \otimes \mathcal{O}_X(S)$  produces a logarithmic Higgs field on  $(E_* \otimes F_*)_0$ . This logarithmic Higgs field on  $(E_* \otimes F_*)_0$  is a parabolic Higgs field on the parabolic vector bundle  $E_* \otimes F_*$ .  $\square$

**Definition 5.** Let  $(E_*, \varphi)$  be an orthogonal or symplectic parabolic bundle. A parabolic Higgs field  $\theta$  on  $E_*$  is said to be *compatible with  $\varphi$*  if the isomorphism  $\tilde{\varphi}$  in (2.2) takes  $\theta$  to the parabolic Higgs field on  $L_* \otimes E_*^*$  induced by  $\theta$  (the Higgs field on  $L_*$  is taken to be the zero section).

We will explain the above definition of a compatible parabolic Higgs field. Consider the pairing  $\hat{\varphi}$  in (2.3). Since a Higgs field  $\theta$  on  $E_*$  is a section of  $\text{End}(E) \otimes \Omega_X(S)$ , for any holomorphic sections  $s$  and  $t$  of  $E$  defined over an open subset  $U \subset X$ , we have

$$\hat{\varphi}_\theta(s, t) := \hat{\varphi}(\theta(s) \otimes t) + \hat{\varphi}(s \otimes \theta(t)) \in \Gamma(U, L \otimes \Omega_X(S)).$$

The Higgs field  $\theta$  is compatible with  $\varphi$  if and only if  $\hat{\varphi}_\theta(s, t) = 0$  for all  $s$  and  $t$ .

**Definition 6.** An *orthogonal (respectively, symplectic) parabolic Higgs bundle*  $(E_*, \varphi, \theta)$  is an orthogonal (respectively, symplectic) parabolic bundle  $(E_*, \varphi)$  together with a parabolic Higgs field  $\theta$  on  $E_*$  which is compatible with  $\varphi$ .

### 3. POLYSTABILITY

Given a parabolic vector bundle  $E_*$  on  $X$  and a holomorphic subbundle  $F$  of the underlying vector bundle  $E$ , we obtain an induced parabolic structure on  $F$  by restricting the quasi-parabolic filtrations and the parabolic weights of  $E$  to  $F$ . Let  $F_*$  be the parabolic vector bundle obtained this way.

**Definition 7.** Let  $(E_*, \theta)$  be a parabolic Higgs bundle on  $X$ .

- (i)  $(E_*, \theta)$  is called *stable* (respectively, *semistable*) if for every subbundle  $F \subset E$  with  $0 < \text{rank}(F) < \text{rank}(E)$  such that  $\theta(F) \subset F \otimes \Omega_X(S)$  (see Definition 3(i)), the inequality

$$\text{par-}\mu(F_*) < \text{par-}\mu(E_*) \quad (\text{respectively, } \text{par-}\mu(F_*) \leq \text{par-}\mu(E_*))$$

holds.

- (ii)  $(E_*, \theta)$  is called *polystable* if it is semistable and isomorphic to a direct sum of stable parabolic Higgs bundles.

Let  $(E_*, \varphi, \theta)$  be an orthogonal or symplectic parabolic Higgs bundle. As before, the holomorphic vector bundle underlying  $E_*$  will be denoted by  $E$ .

**Definition 8.** The orthogonal or symplectic parabolic Higgs bundle  $(E_*, \varphi, \theta)$  will be called *stable* (respectively, *semistable*) if for every isotropic subbundle  $F \subset E$  of positive rank (see (2.4)) such that  $\theta(F) \subset F \otimes \Omega_X(S)$  (see Definition 3(i)), the following condition holds:

$$\text{par-}\mu(F_*) < \text{par-}\mu(E_*) \quad (\text{respectively, } \text{par-}\mu(F_*) \leq \text{par-}\mu(E_*)).$$

Take any parabolic vector bundle  $V_*$  on  $X$ . Using the natural pairing of  $V_*$  with its parabolic dual  $V_*^*$ , the parabolic vector bundle  $V_* \oplus (L_* \otimes V_*^*)$  is equipped with a symplectic as well as an orthogonal form with values in  $L_*$ . To explain this, note that for any finite dimensional complex vector space  $W_0$ , we have

$$(W_0 \oplus W_0^*) \otimes (W_0 \oplus W_0^*) = \bigwedge^2 (W_0 \oplus W_0^*) \oplus \text{Sym}^2(W_0 \oplus W_0^*),$$

and  $\text{id}_{W_0 \oplus W_0^*} \in \text{End}(W_0 \oplus W_0^*) = (W_0 \oplus W_0^*) \otimes (W_0^* \oplus W_0) = (W_0 \oplus W_0^*) \otimes (W_0 \oplus W_0^*)$  projects to a non-degenerate element in both  $\bigwedge^2 (W_0 \oplus W_0^*)$  and  $\text{Sym}^2(W_0 \oplus W_0^*)$ . Both the symplectic and orthogonal forms on  $V_* \oplus (L_* \otimes V_*^*)$  with values in  $L_*$  will be denoted by  $\varphi_{V_*}$ .

Let  $\theta_V$  be a Higgs field on the parabolic vector bundle  $V_*$ . The Higgs field on  $V_*^*$  given by Lemma 4 will be denoted by  $\theta_V^*$ . The zero Higgs field on  $L_*$  and  $\theta_V^*$  together define a Higgs field on  $L_* \otimes V_*^*$  by Lemma 4; this Higgs field on  $L_* \otimes V_*^*$  will be denoted by  $\theta_V^L$ . We note that the parabolic Higgs field  $\theta_V \oplus \theta_V^L$  on  $V_* \oplus (L_* \otimes V_*^*)$  is compatible with  $\varphi_{V_*}$ , so

$$(3.1) \quad (V_* \oplus (L_* \otimes V_*^*), \varphi_{V_*}, \theta_V \oplus \theta_V^L)$$

is an orthogonal or symplectic parabolic Higgs bundle (depending on whether  $\varphi_{V_*}$  is the natural orthogonal or symplectic form).

**Definition 9.** A semistable orthogonal (respectively, symplectic) parabolic Higgs bundle  $(E_*, \varphi, \theta)$  will be called *polystable* if it is a direct sum of finitely many orthogonal (respectively, symplectic) parabolic Higgs bundles

$$(E_*, \varphi, \theta) = \bigoplus_{i=1}^N (E_*^i, \varphi^i, \theta^i),$$

where each  $(E_*^i, \varphi^i, \theta^i)$  is either stable (see Definition 8) or it is of the form

$$(V_* \oplus (L_* \otimes V_*^*), \varphi_{V_*}, \theta_V \oplus \theta_V^L)$$

(see (3.1)) with  $(V_*, \theta_V)$  being a polystable parabolic Higgs bundle.

To compare the above definition with the definition of polystable orthogonal and symplectic parabolic vector bundles (without Higgs structure) given in [BMW11], note that a direct sum of polystable parabolic orthogonal (respectively, symplectic) Higgs bundles of the same parabolic slope is again polystable. Also, for two parabolic vector bundles  $V_*$  and  $W_*$  with  $V$  and  $W$  as the respective underlying vector bundles, we have

$$(\theta_V \oplus \theta_V^L) \oplus (\theta_W \oplus \theta_W^L) = (\theta_{V \oplus W} \oplus \theta_{V \oplus W}^L) \text{ and } \varphi_{V_* \oplus W_*} = \varphi_{V_*} \oplus \varphi_{W_*}.$$

**Proposition 10.** *Let  $(E_*, \varphi, \theta)$  be a polystable orthogonal or symplectic parabolic Higgs bundle. Then the parabolic Higgs bundle  $(E_*, \theta)$  is polystable.*

*Proof.* Let  $E$  be the vector bundle underlying  $E_*$ . We will first show that  $(E_*, \theta)$  is semistable.

Assume that  $(E_*, \theta)$  is not semistable. Let

$$F_* \subset E_*$$

be the unique parabolic subbundle of  $E_*$  of positive rank such that

- $\theta(F) \subset F \otimes \Omega_X(S)$ , where  $F \subset E$  is the subbundle underlying  $F_*$ ,
- $\text{par-}\mu(F_*) \geq \text{par-}\mu(V_*)$  for all parabolic subbundles  $V_* \subset E_*$  with  $\theta(V) \subset V \otimes \Omega_X(S)$ , where  $V$  is the vector bundle underlying  $V_*$ , and
- $\text{rank}(F_*)$  is maximal among all parabolic subbundles of  $E_*$  satisfying the first two conditions.

The quotient bundle  $E/F$  is equipped with a parabolic structure given by the parabolic structure of  $E_*$ , and this parabolic vector bundle is equipped with a Higgs field given by  $\theta$ . If the parabolic Higgs bundle  $E/F$  equipped with these induced structures is not semistable, we may consider the subbundle of it constructed as above using the three conditions. Proceeding inductively, we get a filtration of parabolic subbundles

$$(3.2) \quad 0 = F_*^0 \subset F_* = F_*^1 \subset F_*^2 \subset \cdots \subset F_*^{m-1} \subset F_*^m = E_*$$

such that for all  $i \in [1, m]$ ,

- $\theta(F_i) \subset F_i \otimes \Omega_X(S)$ ,
- $\text{par-}\mu(F_*^i/F_*^{i-1}) \geq \text{par-}\mu(F'_*)$  for every parabolic subbundle  $F'_* \subset E_*/F_*^{i-1}$  preserved by the Higgs field on  $E_*/F_*^{i-1}$  induced by  $\theta$ , and
- $F_*^i/F_*^{i-1}$  is of maximal rank among all parabolic subbundles of  $E_*/F_*^{i-1}$  satisfying the first two conditions.

The filtration in (3.2) is called the *Harder–Narasimhan* filtration for  $(E_*, \theta)$ .

The Higgs field  $\theta$  induces a Higgs field  $\theta'$  on  $L_* \otimes E_*$  using the zero Higgs field on  $L_*$  (see Lemma 4). We note that  $(L_* \otimes E_*, \theta')$  is not semistable because  $(E_*, \theta)$ , which is

isomorphic to it (see Definition 5), is not semistable. Let

$$(3.3) \quad 0 = G_*^0 \subset G_*^1 \subset G_*^2 \subset \cdots \subset G_*^{m-1} \subset G_*^m = L_* \otimes E_*^*$$

be the Harder–Narasimhan filtration for  $(L_* \otimes E_*^*, \theta')$ . From the uniqueness of the Harder–Narasimhan filtration we conclude that

$$(3.4) \quad \tilde{\varphi}(F_*^i) = G_*^i$$

for all  $i$ , where  $\tilde{\varphi}$  is the isomorphism in (2.2).

We put down some properties of the parabolic slope which are straightforward to derive.

- $\text{par-}\mu(W_* \otimes W'_*) = \text{par-}\mu(W_*) + \text{par-}\mu(W'_*)$  for any parabolic vector bundles  $W_*$  and  $W'_*$ ; also,  $\text{par-}\mu(W_*^*) = -\text{par-}\mu(W_*)$ .
- A parabolic Higgs bundle  $(W_*, \beta)$  is semistable (respectively, polystable) if and only if  $W_*^*$  equipped with the Higgs field induced by  $\beta$  is semistable (respectively, polystable). (This follows from the first property.)
- Let  $(M_*, \gamma)$  be a parabolic Higgs line bundle. A parabolic Higgs vector bundle  $(W_*, \beta)$  is semistable (respectively, polystable) if and only if  $W_* \otimes M_*$  equipped with the Higgs field induced by  $\beta$  and  $\gamma$  (see Lemma 4) is semistable (respectively, polystable). (This also follows from the first property.)

From the above properties it follows immediately that the filtration in (3.3) is given by the dual of the filtration in (3.2). This means that

$$(3.5) \quad (L_* \otimes E_*^*)/G_*^{m-i} = L_* \otimes (F_*^i)^*$$

for all  $i \in [1, m]$ .

Combining (3.5) and (3.4) it follows that  $F_*^1$  is an isotropic subbundle of  $E_*$  for the pairing  $\varphi$ , because the composition

$$F_*^1 \xrightarrow{\tilde{\varphi}} G_*^1 \longrightarrow (L_* \otimes E_*^*)/G_*^{m-1}$$

vanishes identically (recall that  $m \geq 2$ ). Since  $F_* = F_*^1$  (see (3.2)) is an isotropic subbundle for the pairing  $\varphi$ , the subbundle  $F_*$  violates the semistability condition for  $(E_*, \varphi, \theta)$ . But we know that  $(E_*, \varphi, \theta)$  is semistable because it is polystable. In view of this contradiction, we conclude that  $(E_*, \theta)$  is semistable.

Assume that  $(E_*, \theta)$  is not polystable.

Consider all parabolic subbundles

$$V_* \subset E_*$$

such that

- $\text{par-}\mu(V_*) = \text{par-}\mu(E_*)$ ,
- $\theta(V) \subset V \otimes \Omega_X(S)$ , where  $V \subset E$  is the subbundle underlying  $V_*$ , and
- the parabolic Higgs bundle defined by  $V_*$  equipped with the Higgs field induced by  $\theta$  is polystable.

Let  $\widehat{V}_*$  be the parabolic subbundle of  $E_*$  generated by all such parabolic subbundles. From the construction of  $\widehat{V}_*$  it follows immediately that

$$\theta(\widehat{V}) \subset \widehat{V} \otimes \Omega_X(S),$$

where  $\widehat{V} \subset E$  is the subbundle underlying  $\widehat{V}_*$ . Let  $\theta_{\widehat{V}}$  be the Higgs field on  $\widehat{V}_*$  defined by  $\theta$ . We have

- $\text{par-}\mu(\widehat{V}_*) = \text{par-}\mu(E_*)$ ,
- the parabolic Higgs bundle  $(\widehat{V}_*, \theta_{\widehat{V}})$  is polystable, and
- $\widehat{V}_*$  is of maximal rank among all parabolic subbundles of  $E_*$  satisfying the first two conditions.

These follow from the fact that for any two parabolic subbundles  $W_*^1$  and  $W_*^2$  of  $E_*$  preserved by  $\theta$  and satisfying the two conditions

- $\text{par-}\mu(W_*^j) = \text{par-}\mu(E_*)$ ,  $j = 1, 2$ , and
- $W_*^j$  equipped with the parabolic Higgs field induced by  $\theta$  is polystable,

the parabolic subbundle of  $E_*$  generated by  $W_*^1$  and  $W_*^2$  is preserved by  $\theta$  and also satisfies the above two conditions. (See [HL97, p. 23, Lemma 1.5.5].)

Since  $(E_*, \theta)$  is assumed to be not polystable, we have  $\text{rank}(\widehat{V}_*) < \text{rank}(E_*)$ .

This polystable parabolic Higgs bundle  $(\widehat{V}_*, \theta_{\widehat{V}})$  is called the *socle* of  $(E_*, \theta)$ . Just as in the case of the Harder–Narasimhan filtration, we get a filtration of parabolic subbundles

$$(3.6) \quad 0 = V_*^0 \subset V_* = V_*^1 \subset V_*^2 \subset \cdots \subset V_*^{n-1} \subset V_*^n = E_*$$

such that for all  $i \in [1, n]$ ,

- $\theta(V_i) \subset V_i \otimes \Omega_X(S)$ , and
- $V_*^i/V_*^{i-1}$  equipped with the Higgs structure induced by  $\theta$  is the socle of  $E_*/V_*^{i-1}$  equipped with the Higgs structure induced by  $\theta$ .

The filtration in (3.6) is called the *socle filtration* for  $(E_*, \theta)$ .

We note that  $(L_* \otimes E_*^*, \theta')$  is semistable because it is isomorphic to the semistable parabolic Higgs bundle  $(E_*, \theta)$ . Let

$$(3.7) \quad 0 = W_*^0 \subset W_*^1 \subset W_*^2 \subset \cdots \subset W_*^{n-1} \subset W_*^n = L_* \otimes E_*^*$$

be the socle filtration for  $(L_* \otimes E_*^*, \theta')$ . From the uniqueness of the socle filtration it follows that

$$(3.8) \quad \widetilde{\varphi}(V_*^i) = W_*^i$$

for all  $i$ , where  $\widetilde{\varphi}$  is the isomorphism in (2.2).

From the properties of the parabolic slope listed above it follows that

$$(L_* \otimes E_*^*)/W_*^{n-i} = L_* \otimes (V_*^i)^*$$

for all  $i \in [1, n]$ . Just as before, this and (3.8) together imply that  $V_*^1 \subset E_*$  is an isotropic subbundle for  $\varphi$ .



Consider  $V_*^1 = V_*$  in (3.6). Let  $\theta_V$  be the Higgs field on  $V_*$  induced by  $\theta$ . Since  $V_* \subset E_*$  is an isotropic subbundle for  $\varphi$  with  $\text{par-}\mu(V_*) = \text{par-}\mu(E_*)$  and preserved by  $\theta$ , from the definition of polystability of  $(E_*, \varphi, \theta)$  it follows that there is an orthogonal or symplectic parabolic Higgs bundle  $(W_*, \phi, \alpha)$  (depending on whether  $(E_*, \varphi)$  is orthogonal or symplectic) such that

$$(3.9) \quad (E_*, \varphi, \theta) = (V_* \oplus (L_* \otimes V_*^*), \varphi_{V_*}, \theta_V \oplus \theta_V^L) \oplus (W_*, \phi, \alpha),$$

where  $(V_* \oplus (L_* \otimes V_*^*), \varphi_{V_*}, \theta_V \oplus \theta_V^L)$  is defined in (3.1).

We have shown that the parabolic Higgs bundle  $(E_*, \theta)$  is semistable. Therefore, from (3.9) it follows that both the parabolic Higgs bundles  $(V_* \oplus (L_* \otimes V_*^*), \theta_V \oplus \theta_V^L)$  and  $(W_*, \alpha)$  are semistable.

We note that the parabolic Higgs bundle  $(L_* \otimes V_*^*, \theta_V^L)$  is polystable, because  $(V_*, \theta_V)$  is polystable. Also, we have  $\text{par-}\mu(L_* \otimes V_*^*) = \text{par-}\mu(E_*)$ , because  $\text{par-}\mu(V_*) = \text{par-}\mu(E_*)$ . Since  $(V_*, \theta_V)$  is the socle of  $(E_*, \theta)$ , these imply that the Higgs parabolic subbundle

$$(L_* \otimes V_*^*, \theta_V^L) \subset (E_*, \theta)$$

is actually contained in  $(V_*, \theta_V)$ . But this contradicts (3.9). Therefore, we conclude that  $(E_*, \theta)$  is polystable.  $\square$

#### 4. HERMITIAN-EINSTEIN CONNECTIONS

Fix a Hermitian metric  $\omega$  on  $X \setminus S$  which extends smoothly over  $X$ ; it is Kähler because  $\dim_{\mathbb{C}} X = 1$ .

**Definition 11.** A *Hermitian-Einstein metric* on a Higgs vector bundle  $(E, \theta)$  over  $X \setminus S$  is defined to be a Hermitian metric  $h$  on  $E$  such that its Chern curvature form  $F_h$  satisfies the equation

$$(4.1) \quad \sqrt{-1} \cdot \Lambda_{\omega}(F_h + [\theta, \theta^*]) = \lambda \cdot \text{id}_E$$

for some  $\lambda \in \mathbb{R}$  which is known as the *Einstein factor*; here,  $\Lambda_{\omega}$  is the adjoint of forming the wedge product with  $\omega$  and  $\theta^*$  is the adjoint endomorphism of  $\theta$  with respect to  $h$ , and  $[\cdot, \cdot]$  is defined using the exterior product on forms and the Lie algebra structure of the fibers of  $\text{End}(E)$ .

If  $h$  is a Hermitian-Einstein metric, then its Chern connection is called a *Hermitian-Einstein connection*.

Let  $(E_*, \theta)$  be a parabolic Higgs bundle. In [Si90, Theorem 4], Simpson describes a construction of a background metric on  $E$  over  $X \setminus S$  from the given data  $(E_*, \theta)$ , which is compatible with taking parabolic duals. Also, it is compatible with taking parabolic tensor products up to mutual boundedness of the resulting background metrics. (See [Si90, Proposition 3.1, Corollary 4.3, Theorem 4].) The metric on  $E$  over  $X \setminus S$  obtained from  $(E_*, \theta)$  via this construction will be denoted by  $h_0(E_*, \theta)$ .

The following existence result is known (see [Si90, Lemma 6.3, Theorem 6], [Si88, Theorem 1]):

**Theorem 12.** *If  $(E_*, \theta)$  is stable, then there is a Hermitian-Einstein metric  $h$  on  $(E, \theta)$  over  $X \setminus S$  such that the metric  $h$  and the background metric  $h_0(E_*, \theta)$  are mutually bounded.*

We will prove the uniqueness of the associated Hermitian-Einstein connection in Theorem 12. For this, we need the following lemma:

**Lemma 13.** *Let  $(F, \tilde{\theta})$  be a Higgs vector bundle on  $X \setminus S$  admitting a Hermitian-Einstein metric  $h$  with Einstein factor  $\lambda = 0$ . Let  $\sigma$  be a holomorphic section of  $F$  satisfying the conditions that  $\tilde{\theta}(\sigma) = 0$  and  $\sigma$  is bounded with respect to  $h$ . Then  $\sigma$  is parallel with respect to the Chern connection  $D$  of  $h$ .*

*Proof.* Let  $\square := \sqrt{-1} \cdot \Lambda_\omega \bar{\partial} \partial$  be the (complex) Laplacian on functions with respect to  $\omega$ . By [Si88, Proposition 2.4], we know that the manifold  $X \setminus S$  satisfies the following condition:

$$(4.2) \quad \begin{array}{l} \text{If } f \text{ is a bounded non-negative smooth function on } X \setminus S, \\ \text{then } \square f \leq 0 \text{ implies } \square f = 0. \end{array}$$

We want to apply (4.2) to the function  $f := |\sigma|_h^2$ . Since  $\sigma$  is holomorphic, we have

$$\square |\sigma|_h^2 = h((\sqrt{-1} \cdot \Lambda_\omega F_h)(\sigma), \sigma) - |D'\sigma|_h^2,$$

where  $D'$  is the  $(1, 0)$  component of the Chern connection  $D$ . As the Einstein factor of  $h$  is 0, the Hermitian-Einstein equation (4.1) implies that

$$\sqrt{-1} \cdot \Lambda_\omega F_h = -\sqrt{-1} \cdot \Lambda_\omega [\tilde{\theta}, \tilde{\theta}^*].$$

Combining this with  $\tilde{\theta}(\sigma) = 0$  it follows that

$$h((\sqrt{-1} \cdot \Lambda_\omega F_h)(\sigma), \sigma) = -\sqrt{-1} \cdot \Lambda_\omega h([\tilde{\theta}, \tilde{\theta}^*](\sigma), \sigma) = -|\tilde{\theta}^*(\sigma)|_h^2 \leq 0.$$

Consequently, we have

$$\square |\sigma|_h^2 \leq -|D'\sigma|_h^2 \leq 0.$$

Since  $|\sigma|_h^2$  is bounded, condition (4.2) yields  $\square |\sigma|_h^2 = 0$ , and thus  $D'\sigma = 0$ . As  $\sigma$  is holomorphic, this already implies that  $D\sigma = 0$ .  $\square$

**Proposition 14.** *Let  $h_1$  and  $h_2$  be two Hermitian-Einstein metrics on  $(E, \theta)$  over  $X \setminus S$  which are mutually bounded. Then the corresponding Chern connections agree. In particular, the Hermitian-Einstein connection on  $(E, \theta)$  over  $X \setminus S$  given by Theorem 12 is unique.*

*Proof.* Let  $F := \text{End}(E) = E \otimes E^*$  be the endomorphism bundle of  $E$  over  $X \setminus S$  equipped with the Higgs field  $\tilde{\theta}$  induced by  $\theta$ . Let  $h$  be the Hermitian metric on  $F$  induced by  $h_1$  and  $h_2$ . Then  $h$  is a Hermitian-Einstein metric on  $F$  with Einstein factor  $\lambda = 0$ . Its Chern connection is

$$D = D_1 \otimes \text{id}_{E^*} + \text{id}_E \otimes D_2^*,$$

where  $D_1$  and  $D_2$  are the Chern connections associated to  $h_1$  and  $h_2$ , respectively, and  $D_2^*$  is the connection on  $E^*$  induced by  $D_2$ .

We want to apply Lemma 13 to the holomorphic section  $\sigma := \text{id}_E$  of  $\text{End}(E)$ . Since  $\text{id}_E$  commutes with  $\theta$ , we have  $\tilde{\theta}(\text{id}_E) = 0$ . The mutual boundedness of  $h_1$  and  $h_2$  implies that  $\text{id}_E$  is bounded with respect to  $h$ . Lemma 13 then yields

$$0 = D(\text{id}_E) = D_1 \circ \text{id}_E - \text{id}_E \circ D_2,$$

and thus  $D_1 = D_2$ .  $\square$

Now let  $(E_*, \varphi, \theta)$  be an orthogonal or symplectic parabolic Higgs bundle on  $X$ . Let  $L_*$  be the parabolic line bundle that we fixed earlier. Since  $L_*$  is stable, it admits a Hermitian-Einstein metric  $h_L$  by Theorem 12 (the Higgs field on  $L_*$  is always taken to be the zero section). The corresponding Hermitian-Einstein connection on  $L_*$  is unique by Proposition 14; denote this connection by  $\nabla_L$ .

**Definition 15.** A *Hermitian-Einstein connection* on  $(E_*, \varphi, \theta)$  is a Hermitian-Einstein connection  $D$  on the underlying Higgs vector bundle  $(E, \theta)$  over  $X \setminus S$  such that the isomorphism  $\tilde{\varphi}$  in (2.2) takes  $D$  to the connection on  $L_* \otimes E_*^*$  induced by  $\nabla_L$  and the dual connection  $D^*$  on  $E_*^*$  for  $D$ .

**Theorem 16.** *Let  $(E_*, \varphi, \theta)$  be a polystable orthogonal or symplectic parabolic Higgs bundle on  $X$ . Then  $(E_*, \varphi, \theta)$  admits a Hermitian-Einstein connection.*

*Proof.* Since  $(E_*, \varphi, \theta)$  is polystable, the parabolic Higgs bundle  $(E_*, \theta)$  is polystable by Proposition 10. By Theorem 12, there is a Hermitian-Einstein metric  $h$  on  $(E, \theta)$  over  $X \setminus S$  such that  $h$  and  $h_0(E_*, \theta)$  are mutually bounded.

We have to show that the Chern connection  $D$  on  $E_*$  associated to  $h$  is a Hermitian-Einstein connection on  $(E_*, \varphi, \theta)$ . By Lemma 4, the Higgs field  $\theta$  induces a Higgs field  $\tilde{\theta}$  on the parabolic vector bundle  $L_* \otimes E_*^*$ . The Hermitian metric  $h'$  on  $L_* \otimes E_*^*$  induced by  $h_L$  and  $h$  is a Hermitian-Einstein metric on  $(L_* \otimes E_*^*, \tilde{\theta})$ . As  $h$  and  $h_0(E_*, \theta)$  are mutually bounded, and the construction of the background metric is compatible with taking duals and tensor products up to mutual boundedness, it follows that  $h'$  and  $h_0(L_* \otimes E_*^*, \tilde{\theta})$  are mutually bounded.

On the other hand, the Hermitian metric  $h''$  on  $L_* \otimes E_*^*$  given by the isomorphism  $\tilde{\varphi}$  in (2.2) is also a Hermitian-Einstein metric on  $(L_* \otimes E_*^*, \tilde{\theta})$  because the Higgs field  $\theta$  is compatible with the orthogonal or symplectic structure  $\varphi$  (see Definition 6). As  $\tilde{\varphi}$  is an isomorphism of parabolic bundles, the metrics  $h''$  and  $h_0(L_* \otimes E_*^*, \tilde{\theta})$  are mutually bounded.

By Proposition 14 it follows that the Chern connections of  $h'$  and  $h''$  coincide. This means that the Chern connection  $D$  associated to  $h$  is a Hermitian-Einstein connection on  $(E_*, \varphi, \theta)$ .  $\square$

There is also a converse to Theorem 16. For this, one has to impose a condition on the asymptotic behavior of the Hermitian-Einstein connection near the parabolic divisor  $S$ . In [Po93], Poritz defines a space

$$(4.3) \quad \mathcal{A} = \mathcal{A}_{\mathcal{D}}^{\delta}$$

of connections depending on the parabolic structure of  $E_*$  (see [Po93, Definition 3.2]). Using this definition, we have:

**Proposition 17.** *Let  $(E_*, \varphi, \theta)$  be an orthogonal or symplectic parabolic Higgs bundle on  $X$ . If  $E$  admits a Hermitian-Einstein connection lying in the space  $\mathcal{A}$ , then it is polystable.*

*Proof.* The proof of [Po93, Theorem 6.4] immediately generalizes to our situation.  $\square$

## REFERENCES

- [BBN01] V. BALAJI, I. BISWAS AND D. S. NAGARAJ: *Principal bundles over projective manifolds with parabolic structure over a divisor*, Tohoku Math. Jour. **53**, 337–367 (2001).
- [BBN03] V. BALAJI, I. BISWAS AND D. S. NAGARAJ: *Ramified  $G$ -bundles as parabolic bundles*, Jour. Ramanujan Math. Soc. **18**, 123–138 (2003).
- [Bi97] I. BISWAS: *Parabolic ample bundles*, Math. Ann. **307**, 511–529 (1997).
- [BMW11] I. BISWAS, S. MAJUMDER AND M. L. WONG: *Orthogonal and symplectic parabolic bundles*, Jour. Geom. Phys. **61**, 1462–1475 (2011).
- [HL97] D. HUYBRECHTS AND M. LEHN: *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997.
- [MY92] M. MARUYAMA AND K. YOKOGAWA: *Moduli of parabolic stable sheaves*, Math. Ann. **293**, 77–99 (1992).
- [Po93] J. A. PORITZ: *Parabolic vector bundles and Hermitian-Yang-Mills connections over a Riemann surface*, Int. Jour. Math. **4**, 467–501 (1993).
- [Se77] C. S. SESHADRI: *Moduli of vector bundles on curves with parabolic structures*, Bull. Am. Math. Soc. **83**, 124–126 (1977).
- [Se82] C. S. SESHADRI: *Fibrés vectoriels sur les courbes algébriques*, in: Astérisque **96**, Société Math. de Fr. (1982).
- [Si88] C. T. SIMPSON: *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, Jour. Am. Math. Soc. **1**, 867–918 (1988).
- [Si90] C. T. SIMPSON: *Harmonic bundles on noncompact curves*, Jour. Am. Math. Soc. **3**, 713–770 (1990).
- [Yo93] K. YOKOGAWA: *Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves*, Jour. Math. Kyoto Univ. **33**, 451–504 (1993).
- [Yo95] K. YOKOGAWA: *Infinitesimal deformation of parabolic Higgs sheaves*, Int. Jour. Math. **6**, 125–148 (1995).

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, BOMBAY 400005, INDIA

*E-mail address:* `indranil@math.tifr.res.in`

*E-mail address:* `stemmler@math.tifr.res.in`